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Pseudo-three-timescale approximation of exponentially modulated free-surface waves

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A novel *pseudo*-three-timescale asymptotic procedure is developed and implemented for obtaining accurate approximations to solutions of an evolution equation arising in thin-film free-surface viscous flow. The new procedure, which employs strained fast *and* slow timescales, requires considerably fewer calculations than its standard three-timescale counterpart employing fast, slow and slower timescales and may readily be applied to other evolution equations of fluid mechanics possessing wave-like solutions exhibiting exponential decay in amplitude and variations in phase over disparate timescales. The new method is validated on the evolution of free-surface waves on a thin, viscous film coating the exterior of a horizontal rotating cylinder and is shown to yield accurate solutions up to non-dimensional times exceeding by an order of magnitude those of previous related studies. Results of the new method applied to this test problem are demonstrated to be in excellent agreement, over large timescales, with those of corroborative spectrally accurate numerical integrations.

1. Introduction

The present work is motivated by the analyses of Hinch & Kelmanson (2003) and Kelmanson (2009), in which two-timescale asymptotics are used to analyse respectively inertia-free and inertial evolution of a viscous free-surface thin film coating the exterior of a circular cylinder rotating about its horizontal axis in a vertical gravitational field; this is the classic problem first modelled by Moffatt (1977) and Pukhnachev (1977). In Hinch & Kelmanson (2003) the fast timescale $\tau = t$ is used, whereas in Kelmanson (2009), it is strained as $\tau = (1+\omega_2\gamma^2+\cdots)t$, where γ is a non-dimensional gravitational parameter, in order to incorporate gravitational drift *a priori*. In both studies, the slow timescale is $T = \gamma^2 t$, so that the asymptotic approximations cease to be uniformly valid when $t = O(\gamma^{-2})$. Because of the slow exponential modulation (decay) on the timescale αT, where α is a non-dimensional surface-tension parameter, it is desirable to extend the duration of unform validity, but the algorithmic cost of implementing a further slow-slow timescale, $\widetilde{T} = \gamma^4 t$ say, is prohibitively expensive because of the rapidly increasing algebraic complexity of the hierarchy of problems at increasing orders of γ.

Accordingly, the idea is proposed in §2 of also straining the *slow* timescale as $T = \gamma^2(\phi_2 + \phi_4\gamma^2 + \cdots)t$ on the physical basis that, for the parameter regime $\gamma^2 \ll \alpha \ll \gamma \ll 1$ considered in previous studies, free-surface waves modes are modulated exponentially by a linear combination of surface tension and inertia, so that the cumulative effect over several timescales is additive in the exponential

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argument. The excellent agreement in § 3 between the results of such 'pseudo-three-timescale' approximations and those of spectrally accurate numerical simulations to times $t \gg O(\gamma^{-2})$ justifies this proposition at a heuristic level, and the absence of justification at a formal level is tempered by the observation of Murdock (1991, p. 245) that even standard three-timescale techniques have 'quite shaky [mathematical] foundations', since the resulting generalized asymptotic expansions do not satisfy uniqueness theorems. An interesting manifestation of such non-uniqueness occurs at the end of § 2, in which two functions and two parameters are determined from only two evolution equations by means of a subtle variation on the secularity-annihilation argument usually associated with multiple-timescale techniques, because the evolution equations are not formally T-secular but rather contain terms of the form $T\mathrm{e}^{-A(\alpha)T}$ for some parameter-dependent $A(\alpha) > 0$.

2. Pseudo-three-timescale asymptotics

Kelmanson (2009) derives the asymptotic evolution equation

$$\partial_t \mathscr{H} + \partial_\theta\{\mathscr{H} - \gamma\mathscr{H}^3\cos\theta + \alpha\mathscr{H}^3\partial_\theta(\partial_\theta{}^2\mathscr{H} + \mathscr{H}) + R\mathscr{H}^3\partial_\theta\mathscr{H}\} = 0, \qquad (2.1)$$

for the non-dimensional 2π-periodic 'perturbed height' $\mathscr{H}(\theta, t)$ of a thin film of viscous liquid adhering to the exterior of a horizontal cylinder of radius a rotating with constant angular velocity ω about its axis in a gravitational field $-g\boldsymbol{j}$. Here θ is the usual polar coordinate centred on the cylinder axis, and the initial condition for (2.1) is $\mathscr{H}(\theta, 0) = 1$. Non-dimensional parameters $\gamma = \rho g\bar{h}^2/3\omega\mu a$, $\alpha = \sigma\bar{h}^3/3\omega\mu a^4$ and $R = (\gamma\rho\sigma^3\bar{h}^{13}/81\alpha^3 g\mu^4 a^{13})^{1/2}$ respectively multiply leading-order terms corresponding to gravity, surface tension and inertia, and μ, ρ, σ and $\bar{h}$ are respectively the dynamic viscosity, density, surface tension and mean dimensional film thickness of the fluid. By Hinch & Kelmanson (2003), the parameter regime

$$\gamma^2 \ll \alpha \ll \gamma \ll 1 \qquad (2.2)$$

is considered. The perturbed height is defined by

$$\mathscr{H}(\theta, t) \equiv \frac{\widetilde{h}(2a + \widetilde{h})}{\bar{h}(2a + \bar{h})}, \qquad (2.3)$$

where $\widetilde{h}(\theta, t) \ll a$ is the dimensional film thickness, so that the evolution equation (2.1) is mass-conserving. If both + signs are replaced by − in (2.3) and if the sign of R is changed in (2.1), the ensuing analysis applies to the interior 'rimming-flow' problem comprehensively studied by Ashmore, Hosoi & Stone (2003). Since inertial effects in the solution of (2.1) are considered in detail in Kelmanson (2009), the Stokes approximation $R = 0$ is now made in (2.1) in order to simplify all subsequent algebra sufficiently to proceed to those orders of γ high enough to emulate the use of three timescales.

With $R = 0$, a solution of (2.1) is sought in the form

$$\mathscr{H}_N(\theta, t) = 1 + \sum_{k=1}^{N}\gamma^k h_k(\theta, \tau, T), \qquad (2.4)$$

in which the h_k are 2π-periodic in θ, $h_k(\theta, 0, 0) = 0$, $k = 1, \ldots, N$, and the fast and slow timescales are both strained as

$$\tau = (1 + \omega_2\gamma^2 + \omega_4\gamma^4 + \cdots)t \equiv \Omega_\tau t \quad \text{and} \quad T = \gamma^2(\phi_2 + \phi_4\gamma^2 + \cdots)t \equiv \Omega_T t, \qquad (2.5)$$

in which, without loss of generality, $\phi_2 \equiv 1$. In (2.5), the fast timescale τ is strained in order to accommodate gravitational drift, and the slow timescale is strained on the heuristic expectation, suggested by the asymptotics of Hinch & Kelmanson (2003), that free-surface wave modes will be modulated on slower and slower timescales by exponential decay, so that their cumulative (multiplicative) modulating effect is additive in the exponential arguments.

Substituting (2.4) into (2.1) follows along the lines similar to those described in Hinch & Kelmanson (2003), the distinction now being the use of two strained timescales (2.5) and the corresponding need to go to higher orders of γ than in the previous study. This is presently achieved via full automation of the solution procedure, using the algebraic manipulator *Maple*, as described in Kelmanson (2009). Accordingly, only novel elements of the solution process are presently described in any detail. At each $O(\gamma^m)$ in the solution tableau, h_m is found to contain terms of the form

$$K_{mn} \equiv (A_{mn}(T)c_{nn} + B_{mn}(T)s_{nn})\,\mathrm{e}^{-n^2(n^2-1)\alpha\tau}, \quad n = m, m-2, \ldots, \tfrac{3+(-1)^m}{2}, \tag{2.6}$$

in which $c_{mn} = \cos(m\theta - n\tau)$, $s_{mn} = \sin(m\theta - n\tau)$ and $A_{mn}(T)$ and $B_{mn}(T)$ are arbitrary functions of T, evolution equations for which arise as secularity conditions in the $O(\gamma^{m+2})$ problem, the γ^2 delay being physically born of a double action of gravity on the wave modes. Note that those terms with (m odd and) $n = 1$ in (2.6) can decay on only the slow timescale T. Solving the $O(\gamma)$ problem yields $A_{11}(0) = -1$ and $B_{11}(0) = 0$, and solving the $O(\gamma^2)$ problem yields $A_{22}(0)$ and $B_{22}(0)$ as rational functions of α. Solving the $O(\gamma^3)$ problem yields $A_{33}(0)$, $B_{33}(0)$, $A_{31}(0)$ and $B_{31}(0)$ as rational functions of α; it also yields the coupled secularity conditions (cf. (3.8) and (3.9) in Hinch & Kelmanson 2003)

$$\partial_T A_{11} = \delta_3 A_{11} + \eta_3 B_{11} \qquad \text{and} \qquad \partial_T B_{11} = -\eta_3 A_{11} + \delta_3 B_{11}, \tag{2.7}$$

in which

$$\delta_3 = -\frac{81\alpha}{1 + 144\alpha^2} \qquad \text{and} \qquad \eta_3 = \omega_2 + \frac{3(5 + 72\alpha^2)}{2(1 + 144\alpha^2)}. \tag{2.8}$$

Using the second equation in (2.8), write $\omega_2 = \epsilon_0 - 3(5 + 72\alpha^2)/(2(1 + 144\alpha^2))$ so that $\eta_3 = \epsilon_0$, from which the initial conditions $A_{11}(0)$ and $B_{11}(0)$ give $B_{11}(T) = 0$ and

$$K_{11} = -\exp(\delta_3 T)\cos(\theta - \tau + \epsilon_0 T).$$

By construction, the straining in τ accommodates gravitational drift (in the fundamental mode, at least), and so $\epsilon_0 = 0$ in K_{11}, giving

$$A_{11} = -\exp(\delta_3 T) \qquad \text{and} \qquad \omega_2 = -\frac{3(5 + 72\alpha^2)}{2(1 + 144\alpha^2)}, \tag{2.9}$$

in exact agreement with Hinch & Kelmanson (2003, equation (3.11)). Although ω_4 and ϕ_4 are yet to be determined, the use of a strained slow timescale T in (2.5), rather than the augmentation of τ and T by a slower timescale, U say, has precluded the introduction of further unknown functions because (2.7) are ordinary differential equations (ODEs) for $A_{11}(T)$ and $B_{11}(T)$ rather than partial differential equations (PDEs) for $A_{11}(T, U)$ and $B_{11}(T, U)$. Hinch & Kelmanson (2003) similarly obtained $A_{11}(T)$ and $B_{11}(T)$ in which $T = \gamma^2 t$ but did not solve for A_{nn} and B_{nn}, $n > 1$, because, by (2.6), these harmonics decay on the fast timescale τ. However, in the present case, explicit forms of the higher-order harmonics are needed to determine ω_4 and ϕ_4.

In the $O(\gamma^4)$ problem, the 2208 terms on the right-hand side, r_4 say, of the PDE for h_4 are reduced to 1058 when the secularity conditions (2.7) are used. Secularity conditions in r_4 in the form of coefficients of c_{11} and s_{11} are precluded by the absence of γ^3 terms in the definition of timescales (2.5). However, secularity conditions do arise in r_4 as the coefficients of $c_{22}\mathrm{e}^{-12\alpha\tau}$ and $s_{22}\mathrm{e}^{-12\alpha\tau}$, yielding slow-timescale evolution equations for A_{22} and B_{22},

$$\partial_T A_{22} = \delta_4 A_{22} - \eta_4 B_{22} \qquad \text{and} \qquad \partial_T B_{22} = \eta_4 A_{22} + \delta_4 B_{22}, \tag{2.10}$$

in which

$$\delta_4 = \frac{54\alpha(360\alpha^2 - 23)}{3600\alpha^2 + 1} + \frac{72\alpha}{5} A_{11}^2 \qquad \text{and} \qquad \eta_4 = \frac{1944\alpha^2(144\alpha^2 + 37)}{(3600\alpha^2 + 1)(144\alpha^2 + 1)}. \tag{2.11}$$

Unlike (2.7), (2.10) has variable coefficients via the A_{11}^2 component of δ_4, and so both A_{22} and B_{22} do not exhibit pure-exponential behaviour on the slow timescale T; rather, they are both products of (different) linear combinations of $\cos \eta_4 T$ and $\sin \eta_4 T$ and a compound-exponential term of the form

$$E_4 \equiv \exp\left(\frac{\nu_1 \exp(\nu_2 T) + \nu_0 T \nu_2 - \nu_1}{\nu_2}\right) \tag{2.12}$$

in which $\nu_0, \nu_2 < 0$ and $\nu_1 > 0$ so that, as $T \to \infty$, $E_4 \to 0$ in both A_{22} and B_{22}. Then the fast-timescale factor $\exp(-12\alpha\tau)$ annihilates K_{22}, allowing h_2 to attain a steady state under the action of surface tension. Details of the construction of the other components of h_2 are omitted. Finally, at this order, initial conditions $A_{44}(0)$, $B_{44}(0)$, $A_{42}(0)$ and $B_{42}(0)$ are generated as (rather complicated) rational functions of α.

In the $O(\gamma^5)$ problem, the 17 862 terms on the right-hand side, r_5 say, of the PDE for h_5 drops to 9586 when the secularity conditions (2.7) and (2.10) are used. Further progress is facilitated† by treating (the reduced) r_5 as a polynomial, in $\beta \equiv \exp(-12\alpha\tau)$, which is found to be of the form

$$r_5 = b_0 + b_1\beta + b_2\beta^2 + b_6\beta^6 + b_7\beta^7 + b_{20}\beta^{20},$$

in which the coefficients are complicated functions of c_{mn}, s_{mn} for $(m, n) \in [1, 5] \times [-5, 5]$, and A_{mn}, B_{mn} for $1 \leqslant m, n \leqslant 4$. Of the 9586 terms in r_5, b_0 (corresponding to pure-harmonic terms) contains 3739; b_1 contains 4010; b_2 contains 631; b_6 contains 934; b_7 contains 128; and b_{20} contains 144. The evolution equations for A_{33} and B_{33} are the coefficients of c_{33} and s_{33} in b_6, because $c_{33}\mathrm{e}^{-72\alpha\tau}$ and $s_{33}\mathrm{e}^{-72\alpha\tau}$ are secular terms in r_5. These evolution equations, together with $A_{33}(0)$ and $B_{33}(0)$ obtained in the $O(\gamma^3)$ problem, reveal that the slow-timescale functions A_{33} and B_{33} also contain compound-exponential behaviour of the generic form given in (2.12).

In order to complete the $O(\gamma^3)$ solution h_3, it still remains to determine ω_4, ϕ_4, A_{31} and B_{31}: it transpires that these are all contained in b_0, which is still so complex that further progress is possible only if the 3739 terms in b_0 are again filtered, this time as a polynomial in A_{11}; then,

$$b_0 = \sum_{j=0}^{4} a_j A_{11}^j$$

† Progress is possibly only if trigonometric terms can be converted into harmonics using *Maple's* `combine` command: this requires judicious and sophisticated recasting of r_5 into suitable alternative forms, since indiscriminate usage of the `combine` command paralyses even moderately powerful computational platforms because of the sheer magnitude and complexity of r_5.

in which a_0, a_1, a_2, a_3 and a_4 respectively contain 414, 972, 1006, 963 and 384 terms. To extract the required evolution equations for A_{31} and B_{31}, it is further necessary to decompose each coefficient a_j into a power series in α. It is the secularities induced by c_{11} and s_{11} in b_0 that respectively yield (decoupled) evolution equations for A_{31} and B_{31},

$$\partial_T A_{31} = \delta_3 A_{31} + f_A(\phi_4, \alpha) A_{11} + g_A(\alpha) A_{11}^3, \tag{2.13}$$

$$\partial_T B_{31} = \delta_3 B_{31} + f_B(\omega_4, \alpha) A_{11} + g_B(\alpha) A_{11}^3, \tag{2.14}$$

in which $\delta_3 < 0$ is given in (2.8), A_{11} is given in (2.9), and f_A, f_B, g_A and g_B are quotients of homogeneous polynomials in α that are all $O(1)$ as $\alpha \to 0$. By the first expression in (2.9), the forcing term $f_A(\phi_4, \alpha)A_{11}$ in (2.13) produces a response in the solution A_{31} of $f_A(\phi_4, \alpha)T \exp(\delta_3 T)$, which attains a maximum of $M = -f_A(\phi_4, \alpha)/(\delta_3 \mathrm{e})$ at $T_M = -1/\delta_3$, so that $M, T_M \to \infty$ as $\alpha \to 0$. Via (2.6), A_{31} has no exponential annihilator on the fast timescale. Thus h_3 is bounded for all $\alpha \ll 1$, the range dictated by (2.2), only if ϕ_4 is a root of $f_A(\phi_4, \alpha)$ for all α, giving

$$\phi_4 = \frac{8545886208\alpha^8 + 444351744\alpha^6 + 4230144\alpha^4 - 75735\alpha^2 + 371}{6(36\alpha^2+1)(1296\alpha^2+1)(144\alpha^2+1)^2}.$$

A parallel argument applied to the solution B_{31} of (2.14) reveals that ω_4 must be a root of $f_B(\omega_4, \alpha)$ for all α, giving

$$\omega_4 = \frac{27(34828517376\alpha^{10} - 8527970304\alpha^8 - 304383744\alpha^6 - 4025376\alpha^4 + 26844\alpha^2 - 25)}{8(36\alpha^2+1)(1296\alpha^2+1)(144\alpha^2+1)^3},$$

in which the limit $\omega_4 \to -675/8$ as $\alpha \to 0$ agrees with (2.6) of Hinch, Kelmanson & Metcalfe (2004). Note that ϕ_4 and ω_4 are determined by making A_{31} and B_{31} bounded functions of α for all $\alpha > 0$, because the ODEs (2.13) and (2.14) are not formally T-secular (Murdock, 1991, p. 248). Finally, both A_{31} and B_{31} are obtained in the form $(C + D\exp(2\delta_3 T))\exp(\delta_3 T)$, so that h_3 is fully determined.

Using the information so far obtained, $\mathscr{H}_3$ can now be constructed. With certain comparisons in §3 in mind, it is also useful to construct $\overline{\mathscr{H}}_3$, in which certain modulation functions A_{mn} and B_{mn} are set to their initial values when, via (2.6), they are annihilated by the action of surface tension on the fast timescale.

Despite the implied complexity of the algebra in this section, the automated computation of $\mathscr{H}_3$ on a modest Dell laptop with a 2.4 GHz Pentium 4 processor and 512 Mb RAM running *Maple 9.0* on Windows XP takes approximately 40 seconds, which time moreover includes several automated checks to ensure that all slow-timescale functions satisfy their respective evolution equations.

3. Results and discussion

The pseudo-three-timescale expansion (P3TE) $\mathscr{H}_3$ obtained in §2 requires full solution of the $O(\gamma^4)$ problem and partial solution of the $O(\gamma^5)$ problem. To obtain a formal three-timescale expansion for $\mathscr{H}_3$ would require full solution of the $O(\gamma^6)$ problem and partial solution of the $O(\gamma^7)$ problem; given the factorial increase in workload at each subsequent order of γ, the advantage of the present approach is manifest. It only remains to verify the accuracy of $\mathscr{H}_3$ by comparison with numerical solutions.

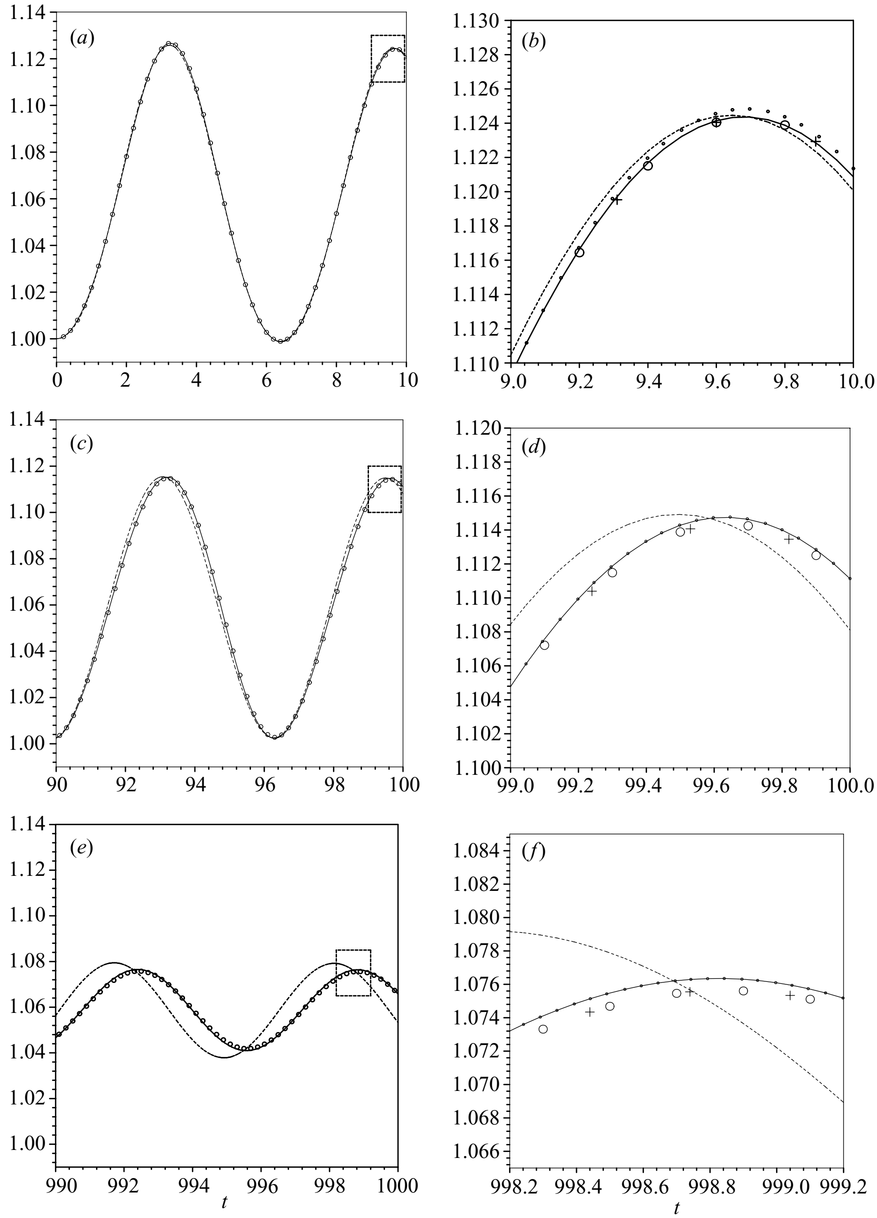

FIGURE 1. Values at the station $\theta = 0$ of the P3TE $\mathscr{H}_3$ (———), the 2TE of Hinch & Kelmanson (2003) (---) and the numerical solution (○) extrapolated from 50-node and 100-node fourth-order finite-difference computations, for parameter values $\gamma = 5.3185335 \times 10^{-2}$ and $\alpha = 4.80710332 \times 10^{-3}$, in time intervals (*a*) $0 \leqslant t \leqslant 10$, (*c*) $90 \leqslant t \leqslant 100$ and (*e*) $990 \leqslant t \leqslant 1000$. Figures (*b*), (*d*) and (*f*) are enlarged views, of the insets in figures (*a*), (*c*) and (*e*) respectively, in which are added both $\overline{\mathscr{H}}_3$ ($\cdots$) and corroborative spectral 32-node numerical results (+) computed by C. M. Groh (2008, private communication) using the fast Fourier transforms (FFTs) of Frigo & Johnson (1998).

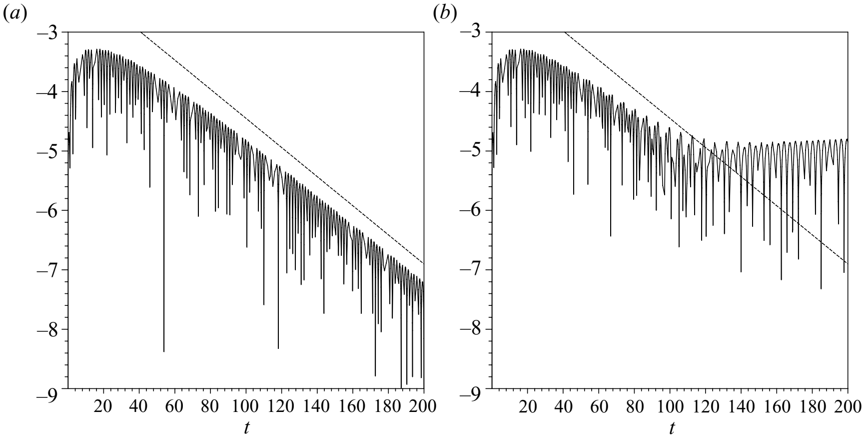

FIGURE 2. Logarithmic (base 10) plots of the absolute difference at the station $\theta = 0$ between the full solution $\mathscr{H}_3$ and reduced solution $\overline{\mathscr{H}}_3$ in which slow-timescale modulation functions $S_{mn}(T)$ are replaced by their initial values $S_{mn}(0)$ for (*a*) $mn = 22$ and $mn = 33$ and (*b*) $mn = 22$, $mn = 33$ and $mn = 31$; (*b*) therefore demonstrates that fixing the slow-timescale components A_{31} and B_{31} at their initial values is mitigated (in the sense of the resulting plateau at a relative error $O(10^{-3})$ per cent) by their premultiplication by γ^3. The dashed straight line has the gradient $-12\alpha\Omega_\tau$, signifying the decay of the first harmonic ($n = 2$) in the exponential argument $-\alpha n^2(n^2 - 1)\tau$ in (2.6).

Figure 1 shows a comparison, at $\theta = 0$ and for $t = O(10)$, $O(10^2)$ and $O(10^3)$, of the P3TE with both the two-timescale expansion (2TE) of Hinch & Kelmanson (2003) and numerical solutions (extrapolated finite-difference and spectral) computed by discretizing the evolution equation (2.1). Parameters γ and α are given the numerical values used in Hinch & Kelmanson (2003) in order to effect the comparison. It is clear that the P3TE represents well both the drift and decay of the modulated free-surface waves and that the duration of validity of this representation is much improved over the 2TE; in particular, figure 1(*e*) shows an appreciable relative drift developing between the 2TE and the numerical results by $t = O(10^3)$.

Also shown in figures 1(*b*), 1(*d*) and 1(*f*) is the reduced PT3E $\overline{\mathscr{H}}_3$, introduced at the end of § 2, in which the slow-timescale functions in the fast-timescale decay terms K_{22} and K_{33} are replaced by their initial values: so rapid is this decay that the difference between $\mathscr{H}_3$ and $\overline{\mathscr{H}}_3$ is perceptible on the presented scale in only figure 1(*b*), at the earliest time $t = O(10)$. A more detailed scrutiny of reducing $\mathscr{H}_3$ to $\overline{\mathscr{H}}_3$ is given in figure 2, which demonstrates the effect of also replacing the slow-timescale components of K_{31} by their initial values in order to simplify matters at $O(\gamma^5)$ in § 2; although by (2.6) there is no fast decay in this case, the relative errors incurred in $\mathscr{H}_3$ are $O(10^{-3})$ per cent, so in practical terms it is arguable that A_{31} and B_{31} need not be computed, their evolution equations having served the purpose of yielding ω_4 and ϕ_4.

Figure 3 demonstrates the P3TE's accurate representation of both decay and drift at large times, specifically $t \approx 2000$ and $t \approx 3000$, by comparison with numerical solutions of (2.1). Because figures 1(*a*), 1(*c*) and 1(*e*) and both graphs in figure 3 have a width of 10 time units, it is possible to compare relative drifts in them; this reveals that the P3TE drift error at $t \approx 3000$ is comparable with the 2TE drift error at $t \approx 100$, quantifying the large extension in uniform validity of the former. In this sense, the heurisitic suggestion of the strained slow timescale in (2.5) is justified, although it is recognized that the present approach may work so well in this case because of the

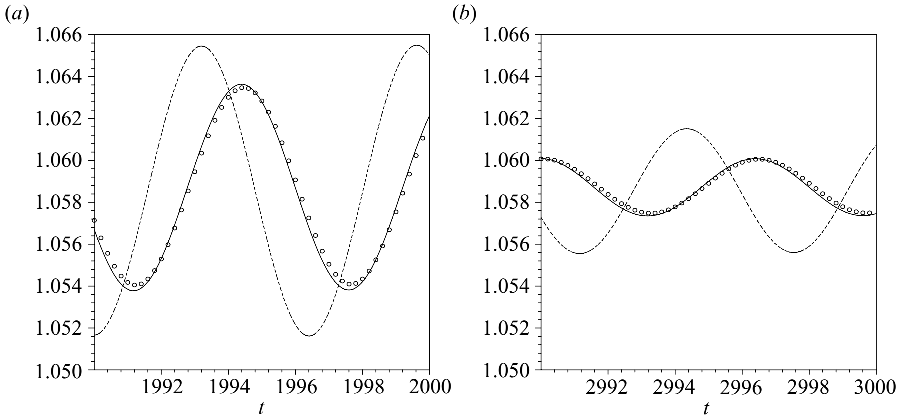

FIGURE 3. Large-time values at the station $\theta = 0$ of the present P3TE $\mathscr{H}_3$ (———), the 2TE of Hinch & Kelmanson (2003) (-----) and the numerical solution (○) from 50-node fourth-order finite-difference computations, for parameter values $\gamma = 5.3185335 \times 10^{-2}$ and $\alpha = 4.80710332 \times 10^{-3}$, in time intervals (*a*) $1990 \leqslant t \leqslant 2000$ and (*b*) $2990 \leqslant t \leqslant 3000$. The vertical scale is 10 times bigger than those in figures 1(*a*), 1(*c*) and 1(*e*). The accurate representation of both the drift and decay at these large times justifies the heurisitic suggestion of the strained slow timescale in (2.5).

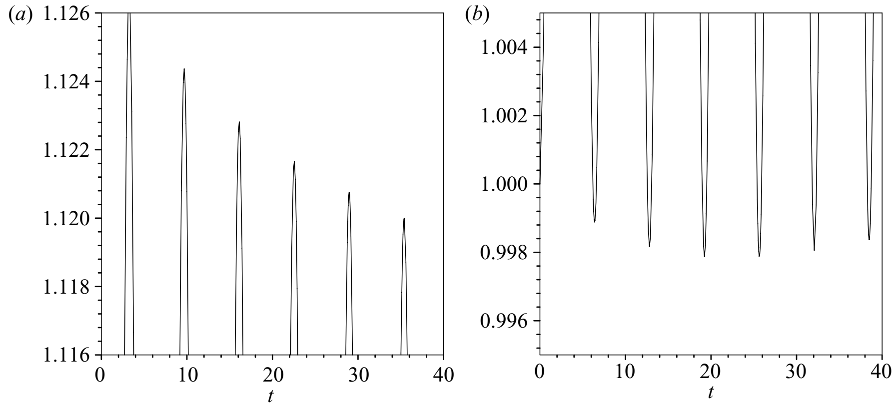

FIGURE 4. Detail of (*a*) maxima and (*b*) minima of $\mathscr{H}_3$ at the station $\theta = 0$ in the interval $0 \leqslant t \leqslant 40$, for $\gamma = 5.3185335 \times 10^{-2}$ and $\alpha = 4.80710332 \times 10^{-3}$.

low-harmonic nature of the solution profile and may not perform equally well in modelling the low-α shock-like solutions in Hinch *et al.* (2004).

Finally, a feature of the maximum and minimum wave heights observed by Hinch & Kelmanson (2003) is revisited and explained. With $\alpha > 0$ it is reasonable to expect that the maximum and minimum elevations will respectively be monotone-decreasing and increasing functions of t. This is not always the case, and an example of counterintuitive behaviour in $\mathscr{H}_3$ is illustrated in figure 4 and also in figures 4–7 of Peterson, Jimack & Kelmanson (2001). Since the same behaviour occurs if only $\mathscr{H}_2$ is used, the *min–min* phenomenon is due to the combination $\mathscr{J} \equiv h_1 + \gamma h_2$. At the station $\theta = 0$, local minima in $\mathscr{J}$ occur at integer multiples of the period $P = 2\pi/\Omega_\tau$, where Ω_τ is defined in (2.5) and determined by ω_2 and ω_4. The time of the minimum minimum may be found by substituting $t = kP$, $k \in \mathbb{N}$, into $\mathscr{J}$ and solving $\partial_k \mathscr{J} = 0$.

Closed-form solution of this equation for k is not possible; expanding it as a series in γ and discarding terms $O(\gamma^3)$ and above is the only truncation that admits an explicit approximation for k, as the solution of

$$\exp(-24\alpha\pi k) \approx \frac{9\gamma(1+36\alpha^2)}{2(1-72\alpha^2)}. \tag{3.1}$$

Although the low order of truncation used in obtaining (3.1) means that its solution, $\widetilde{k}$ say, is not an integer, the time $\widetilde{t}$ of the minimum minimum is approximated by

$$\widetilde{t} = \widetilde{k}P \approx \frac{1}{12\alpha\Omega_\tau} \ln \frac{2(1-72\alpha^2)}{9\gamma(1+36\alpha^2)}, \tag{3.2}$$

in which the reciprocal factor on the right-hand side is identifiable as having originated from the decay of the first harmonic. For the parameters in the caption of figure 4, (3.2) yields $\widetilde{t} \approx 25.3$, which agrees well with the value in the figure. With $\gamma = 5.3185335 \times 10^{-2}$ as in figure 4 and $\alpha = 2.40355166 \times 10^{-3}$, (3.2) yields $\widetilde{t} \approx 129.414$; in this case $\mathscr{H}_3$ has a minimum minimum at $t \approx 132$. When $t = (k+1/2)P$ is similarly substituted into $\mathscr{J}$, the equation corresponding to (3.1) becomes

$$\exp\left(-24\alpha\pi\left(k+\tfrac{1}{2}\right)\right) \approx \frac{9\gamma(1+36\alpha^2)}{2(72\alpha^2-1)}, \tag{3.3}$$

the right-hand side of which is negative, so that there is no corresponding *max–max* phenomenon, at least in the parameter range (2.2) in which the present theory is valid. However, in the low-α limit, where the alternative theory of Hinch *et al.* (2004) is valid, figure 6 of Peterson *et al.* (2001) confirms the existence of a max–max phenomenon.

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